Wave propagation in a generalized thermo elastic plate embedded in elastic medium

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Abstract. In this paper, the wave propagation in a generalized thermo elastic plate embedded in an elastic medium (Winkler model) is studied based on the Lord-Schulman (LS) and Green-Lindsay (GL) generalized two dimensional theory of thermo elasticity. Two displacement potential functions are introduced to uncouple the equations of motion. The frequency equations that include the interaction between the plate and foundation are obtained by the traction free boundary conditions using the Bessel function solutions. The numerical calculations are carried out for the material Zinc and the computed non-dimensional frequency and attenuation coefficient are plotted as the dispersion curves for the plate with thermally insulated and isothermal boundaries. The wave characteristics are found to be more stable and realistic in the presence of thermal relaxation times and the foundation parameter. A comparison of the results for the case with no thermal effects shows well agreement with those by the membrane theory.

Keywords: wave propagation; vibration of thermal plate; plate immersed in fluid; generalized thermo elastic plate; Winkler foundation.

1. Introduction

Cylindrical thin plate plays a vital role in many engineering fields such as aerospace, civil, chemical, mechanical, naval and nuclear engineering. The dynamical interaction between the cylindrical plate and solid foundation has potential applications in modern engineering fields due to the fact that their static and dynamic behaviors will be affected by the surrounding media. The analysis of thermally induced wave propagation of a cylindrical plate embedded in an elastic medium is a problem that may be encountered in the design of structures such as atomic reactors, steam turbines, submarine structures subjected to wave loadings, or for the impact loadings due to superfast trains, or for jets and other devices operating at elevated temperatures. Moreover, it is recognized that the thermal effects on the elastic wave propagation supported by elastic foundations may have implications related to many seismological applications. This study can be potentially used in applications involving nondestructive testing (NDT) and qualitative nondestructive evaluation (QNDE).

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The generalized theory of thermo elasticity was developed by Lord and Schulman (1967), which involves one relaxation time for isotropic homogeneous media, and is called the first generalization to the coupled theory of elasticity. Their equations determine the finite speed of wave propagation of heat and the displacement distributions. The corresponding equations for an isotropic case were obtained by Dhaliwal and Sherief (1980). The second generalization to the coupled theory of elasticity is known as the theory of thermo elasticity with two relaxation times, or as the theory of temperature-dependent thermoelectricity. A generalization of this inequality was proposed by Green and Laws (1972). Green and Lindsay (1972) obtained an explicit version of the constitutive equations. These equations were also obtained independently by Suhubi (1975). This theory contains two constants that act as the relaxation times and modifies not only the heat equations, but also all the equations of the coupled theory. The classical Fourier's law of heat conduction is not violated if the medium under consideration has a center of symmetry. Erbay and Suhubi (1986) studied the longitudinal wave propagation in a generalized thermoplastic infinite cylinder and obtained the dispersion relation for the cylinder with a constant surface temperature. Ponnusamy (2007) has studied wave propagations in a generalized thermo elastic solid cylinder of arbitrary cross sections using the Fourier expansion collocation method. Later, Ponnusamy and Selvamani (2011) obtained mathematical modeling and analysis for a thermo elastic cylindrical panel using the wave propagation approach.

Sharma and Pathania (2005) investigated the generalized wave propagation in circumferential curved plates. Modeling of circumferential waves in a cylindrical thermo elastic plate with voids was discussed by Sharma and Kaur (2010). Ashida and Tauchert (2001) presented the temperature and stress analysis of an elastic circular cylinder in contact with heated rigid stamps. Later, Ashida (2003) analyzed the thermally induced wave propagation in a piezoelectric plate. Tso and Hansen (1995) studied the wave propagation through cylinder/plate junctions. Heyliger and Ramirez (2000) analyzed the free vibration characteristics of laminated circular piezoelectric plates and discs by using a discrete-layer model of the weak form of the equations of periodic motion. The thermal deflection of an inverse thermo elastic problem in a thin isotropic circular plate was presented by Gaikward and Deshmukh (1979). The study about a plate embedded in an elastic medium is important for design of structures such as atomic reactors, steam turbines, submarine structures with wave loads, or for the impact effects due to superfast train, or for jets and other devices operating at elevated temperatures. Selvadurai (2005) has presented the most general form of a soil model used in practical applications. Kamal (1983) discussed a circular plate embedded in an elastic medium, in which the governing differential equation was formulated using the Chebyshev-Lanczos techniques. Paliwal (1996) presented an investigation on the coupled free vibrations of an isotropic circular cylindrical shell on Winkler and Pasternak foundations by employing a membrane theory. The vibration of a circular plate laterally supported by an elastic foundation was investigated by Leissa (1981), which indicates that the effect of the Winkler foundation merely increases the square of the natural frequency of the plate by a constant. Bernhard (1999) studied the buckling frequency for a clamped plate embedded in an elastic medium. Recently, Wang (2005) studied the fundamental frequency of a circular plate supported by a partial elastic foundation using the finite element method.

In this paper, the in-plane vibration of a generalized thermo elastic thin plate embedded in an elastic medium composed of homogeneous isotropic material is studied. The solutions to the equations of motion for an isotropic medium is obtained by using the two dimensional theory of elasticity and Bessel function solutions. The numerical calculations are carried out for the material

Zinc. The computed non-dimensional frequency and attenuation coefficient are plotted as dispersion curves for the plate with thermally insulated and isothermal boundaries.

2. Formulation of the problem

We consider a thin homogeneous, isotropic, thermally conducting elastic plate of radius R with uniform thickness d and temperature T_0 in the undisturbed state initially, totally embedded in a two-parameter elastic medium with the spring layer K and shear layer G is shown in Fig. 1. The system displacements and stresses are defined in the polar coordinates r and θ for an arbitrary point inside the plate, with u denoting the displacement in the radial direction of r and v the displacement in the tangential direction of θ. The in-plane vibration and displacements of the plate embedded in the elastic medium is obtained by assuming that there is no vibration and a displacement along the z axis (normal to the plate) in the cylindrical coordinate system (r, θ, z).

The two dimensional stress equations of motion and heat conduction equation in the absence of body force for a linearly elastic medium are

$$\sigma_{rr,r} + r^{-1}\sigma_{r\theta,\theta} + r^{-1}(\sigma_{rr} - \sigma_{\theta\theta}) = \rho u_{r,tt}$$

$$\sigma_{r\theta,r} + r^{-1}\sigma_{\theta\theta,\theta} + 2r^{-1}\sigma_{r\theta} = \rho u_{\theta,tt}$$

$$k(T_{,rr} + r^{-1}T_{,r} + r^{-2}T_{,\theta\theta}) - \rho c_v(T + \tau_0 T_{,tt}) = \beta T_0\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)[e_{rr} + e_{\theta\theta}] \tag{1}$$

where ρ is the mass density, c_v is the specific heat capacity, $\kappa = k/\rho c_v$ is the diffusivity, k is the thermal conductivity, τ_0 is a thermal relaxation time, and T_0 is the reference temperature. The strain-displacement relations for the plate are

$$\sigma_{rr} = \lambda(e_{rr} + e_{\theta\theta}) + 2\mu e_{rr} - \beta(T + \delta_{2k}\tau_1 T_{,t})$$

$$\sigma_{\theta\theta} = \lambda(e_{rr} + e_{\theta\theta}) + 2\mu e_{\theta\theta} - \beta(T + \delta_{2k}\tau_1 T_{,t})$$

$$\sigma_{r\theta} = 2\mu e_{r\theta} \tag{2}$$

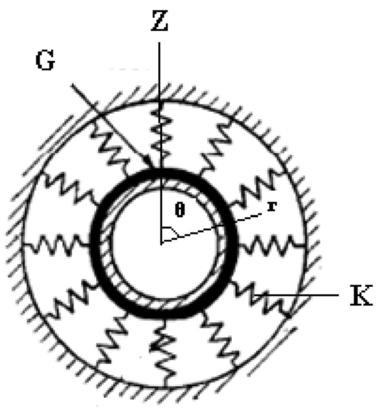

Fig. 1 Geometry of the problem

where e_{ij} are the strain components, $\beta = (3\lambda + 2\mu)\alpha_T$ is the thermal stress coefficients, α_T is the coefficient of linear thermal expansion, T is the temperature, t is time, λ and μ are Lame' constants, τ_1 is a thermal relaxation time, and the comma in the subscripts denotes the partial differentiation with respect to the variable following. Here δ_{ij} is the Kronecker delta function. In addition, we can replace $k = 1$ for the LS theory and $k = 2$ for the GL theory. The thermal relaxation times τ_0 and τ_1 satisfies the inequalities $\tau_0 \geq \tau_1 \geq 0$ for the GL theory only.

The strain e_{ij} are related to the displacements as given by

$$e_{rr} = u_{,r}, \qquad e_{\theta\theta} = r^{-1}(u + v_{,\theta}), \qquad e_{r\theta} = v_{,r} - r^{-1}(v - u_{,\theta}) \tag{3}$$

in which u and v are the displacement components along the radial and circumferential directions, respectively. σ_{rr} and $\sigma_{\theta\theta}$ are the normal stress components and $\sigma_{r\theta}$, $\sigma_{\theta z}$ and σ_{zr} the shear stress components, e_{rr}, $e_{\theta\theta}$ and e_{zz} the normal strain components, and $e_{r\theta}$, $e_{\theta z}$ and e_{zr} the shear strain components.

By substituting Eqs. (3) and (2) into Eq. (1), the following displacement equations of motions are obtained

$$(\lambda + 2\mu)(u_{,rr} + r^{-1}u_{,r} - r^{-2}u) + \mu r^{-2}u_{,\theta\theta} + r^{-1}(\lambda + \mu)v_{,r\theta}$$

$$+ r^{-2}(\lambda + 3\mu)v_{,\theta} - \beta(T_{,r} + T\delta_{2k}\tau_1 T_{,rt}) = \rho u_{,tt}$$

$$\mu(v_{,rr} + r^{-1}v_{,r} - r^{-2}v) + r^{-2}(\lambda + 2\mu)v_{,\theta\theta} + r^{-2}(\lambda + 3\mu)u_{,\theta}$$

$$+ r^{-1}(\lambda + \mu)u_{,r\theta} - -\beta(T_{\theta} + \eta T_{,\theta t}) = \rho v_{,tt}$$

$$k(T_{,rr} + r^{-1}T_{,r} + r^{-2}T_{,\theta\theta}) - \rho c_v(T + \tau_0 T_{,tt}) = \beta T_0\left(\frac{\partial}{\partial t} + \tau_0\delta_{1k}\frac{\partial^2}{\partial t^2}\right)[u_{,r} + r^{-1}(u + v_{,\theta})] \tag{4}$$

The above coupled partial differential equations are also subjected to the following non-dimensional boundary conditions at the surfaces $r = a, b$.

(1) Stress free boundary (Unclamped edge)

$$\sigma_{rr} = \sigma_{r\theta} = 0 \tag{5a}$$

(2) Rigidly fixed boundary (Clamped edge)

$$u = v = 0 \tag{5b}$$

(3) Thermal boundary

$$T_{,r} + hT = 0 \tag{5c}$$

where h is the surface heat transfer coefficient. Here $h \to 0$ corresponds to a thermally insulated surface and $h \to \infty$ refers to an isothermal one.

2.1 Lord-Schulman (LS) theory

Based on the Lord-Schulman theory of thermo elasticity, the three dimensional rate dependent temperature with one relaxation time is obtained by replacing $k = 1$ in the heat conduction equation of Eq. (1), namely,

$$k\left(T_{,rr} + \frac{1}{r}T_{,r} + \frac{1}{r^2}T_{,\theta\theta}\right) = \rho C_v[T + \tau_0 T_{,tt}] + \beta T_0\left[\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right](e_{rr} + e_{\theta\theta}) \tag{6a}$$

The stress-strain relation is replaced by

$$\begin{aligned}\sigma_{rr} &= \lambda(e_{rr} + e_{\theta\theta}) + 2\mu e_{\theta\theta} - \beta(T)\\ \sigma_{\theta\theta} &= \lambda(e_{rr} + e_{\theta\theta}) + 2\mu e_{\theta\theta} - \beta(T)\\ \sigma_{r\theta} &= 2\mu e_{r\theta}\end{aligned} \tag{6b}$$

By substituting the preceding stress-strain relations into Eq. (1), we can get the following displacement equation

$$(\lambda + 2\mu)(u_{,rr} + r^{-1}u_{,r} - r^{-2}u) + r^{-2}\mu u_{,\theta\theta} + r^{-1}(\lambda + \mu)v_{,r\theta} + r^{-2}(\lambda + 3\mu)v_{,\theta} - \beta(T) = \rho u_{,tt}$$

$$\begin{aligned}(\mu)(v_{,rr} + r^{-1}v_{,r} - r^{-2}v) + r^{-1}(\lambda + \mu)u_{,r\theta} + r^{-2}(\lambda + 3\mu)u_{,\theta} + r^{-2}(\lambda + 3\mu)v_{,\theta}\\ + r^{-2}(\lambda + 2\mu)v_{,\theta\theta} - \beta(T) = \rho v_{,tt}\end{aligned} \tag{6c}$$

The symbols and notations involved have the same meanings as defined in earlier sections. Since the heat conduction equation of this theory is of the hyperbolic wave type, it can automatically ensure the finite speeds of propagation for heat and elastic waves.

2.2 Green-Lindsay (GL) theory

The second generalization to the coupled thermo elasticity with two relaxation times called the Green-Lindsay theory of thermo elasticity is obtained by setting $k = 2$ in the heat conduction equation of Eq. (1), namely,

$$k\left(T_{,rr} + \frac{1}{r}T_{,r} + \frac{1}{r^2}T_{,\theta\theta}\right) = \rho C_v[T + \tau_0 T_{,tt}] + \beta T_0\frac{\partial}{\partial t}(e_{rr} + e_{\theta\theta}) \tag{7a}$$

The stress-strain relation is replaced by

$$\begin{aligned}\sigma_{rr} &= \lambda(e_{rr} + e_{\theta\theta}) + 2\mu e_{\theta\theta} - \beta(T + \tau_1 T_{,t})\\ \sigma_{\theta\theta} &= \lambda(e_{rr} + e_{\theta\theta}) + 2\mu e_{\theta\theta} - \beta(T + \tau_1 T_{,t})\\ \sigma_{r\theta} &= 2\mu e_{r\theta}\end{aligned} \tag{7b}$$

By substituting the preceding relations into Eq. (1), the displacement equation can be reduced as

$$(\lambda+2\mu)(u_{,rr}+r^{-1}u_{,r}-r^{-2}u)+r^{-2}\mu u_{,\theta\theta}+r^{-1}(\lambda+\mu)v_{,r\theta}+r^{-2}(\lambda+3\mu)v_{,\theta}-\beta(T_{,r}+\tau_1 T_{,rt}) = \rho u_{,tt}$$

$$(\mu)(v_{,rr}+r^{-1}v_{,r}-r^{-2}v)+r^{-1}(\lambda+\mu)u_{,r\theta}+r^{-2}(\lambda+3\mu)u_{,\theta}+r^{-2}(\lambda+3\mu)v_{,\theta}$$

$$+r^{-2}(\lambda+2\mu)v_{,\theta\theta}-\beta(T_{,\theta}+\tau_1 T_{,\theta t}) = \rho v_{,tt} \quad (7c)$$

where the symbols and notations have been defined in the previous sections. In view of available experimental evidence in favor of the finiteness of heat propagation speeds, the generalized thermo elasticity theories are considered to be more realistic than the conventional theory in dealing with practical problems involving very large heat fluxes and/or short time intervals, such as those occurring in laser units and energy channels.

To uncouple Eq. (7), the mechanical displacement u, v along the radial and circumferential directions given by Sharma (2010) are adopted as follows

$$u = \phi_{,r}+r^{-1}\psi_{,\theta} \qquad v = r^{-1}\phi_{,\theta}-\psi_{,r} \quad (8)$$

Substituting Eq. (8) into Eq. (7) yields the following second order partial differential equation with constant coefficients

$$\{(\lambda+2\mu)\nabla^2+\rho\omega^2\}\phi-\beta(T+\delta_{2k}\tau_1 T_{,t}) = 0 \quad (9a)$$

$$\{k\nabla^2-\rho C_{\upsilon}i\omega\eta_0\}T+\beta T_0(i\omega\eta_1)\nabla^2\phi = 0 \quad (9b)$$

$$\left(\nabla^2+\frac{\rho}{\mu}\omega^2\right)\psi = 0 \quad (9c)$$

where $\nabla^2 \equiv \partial^2/\partial x^2+x^{-1}\partial/\partial x+x^{-2}\partial^2/\partial\theta^2$.

3. Solutions of the problem

The equations are given in Eq. (9) are coupled partial differential equations with two displacements and heat conduction components. To uncouple these equations, we assume the vibration and displacements along the axial direction z to be zero. Hence, the solutions of Eq. (9) can be presented in the following form

$$\phi(r,\theta,t) = \bar{\phi}(r)\exp\{i(p\theta-\omega t)\} \quad (10a)$$

$$\psi(r,\theta,t) = \bar{\psi}(r)\exp\{i(p\theta-\omega t)\} \quad (10b)$$

$$T(r,\theta,t) = (\lambda+2\mu/\beta\alpha^2)\bar{T}(r)\exp\{i(p\theta-\omega t)\} \quad (10c)$$

where $i = \sqrt{-1}$, ω is the angular frequency, p is the angular wave number, $\phi(r,\theta)$, $\psi(r,\theta)$ and $T(r,\theta)$ are the displacement potentials. Substituting Eq. (10) into Eq. (9) and introducing the dimensionless quantities such as $x = r/a$, $c_1^2 = (\lambda+2\mu)/\rho$, $c_2^2 = \mu/\rho\Omega^2 = \rho\omega^2a^2/\mu$, $\bar{\lambda} =$

λ/μ $\bar{d} = \rho c_v \mu/\beta T_0$, we can get the following partial differential equation with constant coefficients

$$\{(2+\bar{\lambda})\nabla_1^2 + \Omega^2\}\phi - (2+\bar{\lambda})\eta_2 T = 0 \tag{11a}$$

$$\{k_1\nabla_1^2 - i\omega\bar{d}\eta_0\}T + \beta T_0(i\omega\eta_1)\nabla_1^2\phi = 0 \tag{11b}$$

$$\text{and } (\nabla_1^2 + \Omega^2)\psi = 0 \tag{11c}$$

where $\nabla_2^2 = \frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r} - \frac{p^2}{r^2}$ and $\eta_0 = 1 + i\omega\tau_0$ $\eta_0 = 1 + i\omega\tau_0$, $\eta_1 = 1 + i\omega\delta_{1k}\tau_0$, $\eta_2 = 1 +$ iω

$\delta_{2k}\tau_1$ Eq. (11c) in terms of ψ gives a purely transverse wave. This wave is polarized in planes perpendicular to the z-axis. We assume that the disturbance is time harmonic through the factor $e^{i\omega t}$. Rewriting Eq. (11) yields the following fourth order differential equation

$$(A\nabla_2^4 + B\nabla_2^2 + C)\phi = 0 \tag{12}$$

where $A = (2+\bar{\lambda})k_1$, $B = \{k_1\Omega^2 - i\omega(2+\bar{\lambda})\bar{d}\eta_0 + i\omega T_0(2+\bar{\lambda})\beta\eta_1\eta_2\}$, $c = -(i\omega\Omega^2\bar{d}\eta_0)$

By solving the partial differential Eq. (10), the solutions is obtained as

$$\bar{\phi} = \sum_{i=1}^{2}[A_iJ_n(\alpha_i ax) + Y_nB_i(\alpha_i ax)]\cos n\theta \tag{13a}$$

$$\bar{T} = \sum_{i=1}^{2}d_i[A_iJ_n(\alpha_i ax) + Y_nB_i(\alpha_i ax)]\cos n\theta \tag{13b}$$

$$\text{where } d_i = \{k_1(\alpha_i ax)^4 + (2+\bar{\lambda})\beta T_0 i\omega\eta_1\eta_2(\alpha_i ax)^2 - (2+\bar{\lambda})i\omega\bar{d} \tag{14}$$

Eq. (11c) is a Bessel equation with possible solutions given as

$$\bar{\psi} = \begin{cases} A_3J_n(\alpha_3 ax) + B_3Y_n(\alpha_3 ax) & \alpha_3 ax > 0 \\ A_3a^n + B_3a^{-n} & \alpha_3 ax = 0 \\ A_3I_n(\alpha_3 ax) + B_3K_n(\alpha_3 ax) & \alpha_3 ax < 0 \end{cases} \tag{15}$$

where J_n and Y_n are Bessel functions of the first and second kinds, respectively, while I_n and k_n are modified Bessel functions of first and second kinds, respectively. (A_i, B_i) $i = 1, 2, 3$ are arbitrary constants. Since $\alpha_3 ax \neq 0$, thus the condition $\alpha_3 ax \neq 0$ will not be discussed in the following. For convenience, we will pay attention only to the case of $\alpha_3 ax > 0$ in what follows. The derivation for the case of $\alpha_3 ax < 0$ is similar.

$$\bar{\psi} = [A_3J_n(\alpha_3 ax) + B_3(\alpha_3 ax)]\sin n\theta \tag{16}$$

where $(\alpha_3 a)^2 = \Omega^2$.

4. Boundary conditions and frequency equations

In this section we shall derive the frequency equation for the three dimensional vibration of the cylindrical panel subjected to stress free boundary conditions at the upper and lower surfaces at $r = a, b$. Substituting the expressions in Eqs. (1)-(3) into Eq. (5), we can get the frequency equation for free vibration as follows

$$|E_{ij}^1| = 0 \qquad i,j = 1,2\ldots\ldots6 \tag{17}$$

$$E_{11}^1 = (2+\bar{\lambda})((nJ_n(\alpha_1 ax) + (\alpha_1 ax)J_{n+1}(\alpha_1 ax)) - ((\alpha_1 ax)^2 R^2 - n^2)J_n(\alpha_1 ax))$$
$$+\bar{\lambda}(n(n-1)(J_n(\alpha_1 ax) - (\alpha_1 ax)J_{\delta+1}(\alpha_1 ax))) - \beta T(i\omega)\eta_2 d_1(\alpha_1 ax)^2$$

$$E_{13}^1 = (2+\bar{\lambda})((nJ_n(\alpha_2 ax) + (\alpha_2 ax)J_{n+1}(\alpha_2 ax)) - ((\alpha_2 ax)^2 R^2 - n^2)J_n(\alpha_2 ax))$$
$$+\bar{\lambda}(n(n-1)(J_n(\alpha_2 ax) - (\alpha_2 ax)J_{\delta+1}(\alpha_2 ax))) - \beta T(i\omega)\eta_2 d_2(\alpha_2 ax)^2$$

$$E_{15}^1 = (2+\bar{\lambda})((n(n-1)J_n(\alpha_3 ax) - (\alpha_3 ax)J_{n+1}(\alpha_3 ax)) + \bar{\lambda}(n(n-1)J_n(\alpha_3 ax) - (\alpha_3 ax)J_{n+1}(\alpha_3 ax))$$

$$E_{21}^1 = 2n(n-1)J_n(\alpha_1 ax) - 2n(\alpha_1 ax)J_{n+1}(\alpha_1 ax)$$

$$E_{23}^1 = 2n(n-1)J_n(\alpha_2 ax) - 2n(\alpha_2 ax)J_{n+1}(\alpha_2 ax)$$

$$E_{25}^1 = 2n(n-1)J_n(\alpha_3 ax) - 2(\alpha_3 ax)J_{\delta+1}(\alpha_3 ax) + ((\alpha_3 ax)^2 - n^2)J_n(\alpha_3 ax))$$

$$E_{31}^1 = d_1(nJ_n(\alpha_1 ax) - (\alpha_1 ax)J_{n+1}(\alpha_1 ax) + hJ_n(\alpha_1 ax))$$

$$E_{33}^1 = d_2(nJ_n(\alpha_2 ax) - (\alpha_2 ax)J_{n+1}(\alpha_2 ax) + hJ_n(\alpha_2 ax))$$

$$E_{35}^1 = 0$$

Obviously E_{ij} $(j = 2, 4, 6)$ can be obtained by just replacing the Bessel functions of the first kind in E_{ij} $(i = 1, 3, 5)$ with those of the second kind, respectively, while E_{ij} $(i = 4, 5, 6)$ can be obtained by just replacing a in E_{ij} $(i = 1, 2, 3)$ with b. Now we consider the coupled free vibration problem. Allowing for the effect of the surrounded elastic medium, which is treated as the Pasternak model, the boundary conditions at the inner and outer surfaces $r = a$, b can considered as follows

$$\sigma_{rr} = \sigma_{r\theta} = 0, \qquad T_{,r} = 0 \qquad (r = a) \tag{18}$$

$$\sigma_{rr} = -Ku + G\Delta u \qquad \sigma_{r\theta} = 0 \qquad (r = b) \tag{19}$$

where $\Delta = \partial^2/\partial z^2 + (1/r^2)\partial^2/\partial\theta^2$, K is the foundation modulus and G is the shear modulus of the foundation. It is mentioned here that the elastic medium can be modeled as the Winkler type by setting $G = 0$ in Eq. (19). From Eqs. (18)-(19) and the results obtained in the preceding section, we get the coupled free vibration frequency equation as follows

$$\left|E_{ij}^2\right| = 0 \qquad i,j = 1,2\ldots\ldots6 \tag{20}$$

$$E_{ij}^2 = E_{ij}^1(i = 1, 2, 3, 4, 6; j = 1, 2\ldots6)$$

$$E_{41}^2 = E_{41}^1 - p^*(nJ_n(\alpha_1 ax) - (\alpha_1 ax)J_{n+1}(\alpha_1 ax))$$

$$E_{42}^2 = E_{42}^1 - p^*(nY_n(\alpha_1 ax) - (\alpha_1 ax)Y_{n+1}(\alpha_1 ax))$$

$$E_{43}^2 = E_{43}^1 - p^*(nJ_n(\alpha_2 ax) - (\alpha_2 ax)J_{n+1}(\alpha_2 ax))$$

$$E_{44}^2 = E_{44}^1 - p^*(nY_n(\alpha_2 ax) - (\alpha_2 ax)Y_{n+1}(\alpha_2 ax))$$

$$E_{45}^2 = E_{45}^1 - p^*n(nJ_n(\alpha_3 ax) - (\alpha_3 ax)J_{n+1}(\alpha_3 ax))$$

$$E_{46}^2 = E_{46}^1 - p^*n(nY_n(\alpha_3 ax) - (\alpha_3 ax)Y_{n+1}(\alpha_3 ax))$$

$$p^* = p_1 + p_2(p^2 + n^2) \quad \text{where} \quad p_1 = KR/\mu \quad \text{and} \quad p_2 = G/R\mu$$

5. Numerical results and discussion

The coupled free wave propagation in a simply supported homogenous isotropic thermo elastic cylindrical plate embedded in a Winkler type of elastic medium is numerically solved for the Zinc material by setting $p_2 = 0$ and Winkler elastic modulus $K = 1.5 \times 10^7$. The material properties of Zinc are given as follows

$$\rho = 7.14 \times 10^3 kgm^{-3} \quad T_0 = 296K \quad K = 1.24 \times 10^2 Wm^{-1}\deg^{-1}$$

$$\mu = 0.508 \times 10^{11} Nm^{-2} \quad \beta = 5.75 \times 10^6 Nm^{-2}\deg^{-1} \quad \in_1 = 0.0221$$

$$\lambda = 0.385 \times 10^{11} Nm^{-2} \quad \text{and} \quad C_v = 3.9 \times 10^2 Jkg^{-1}\deg^{-1}$$

The roots of the algebraic equation in Eq. (12) were calculated using a combination of the Birge-Vita method and Newton-Raphson method. For the present case, the simple Birge-Vita method does not work for finding the root of the algebraic equation. After obtaining the roots of the algebraic equation using the Birge-Vita method, the roots are corrected for the desired accuracy using the Newton-Raphson method. Such a combination can overcome the difficulties encountered in finding the roots of the algebraic equations of the governing equations. Here the values of the thermal relaxation times are calculated from Chandrasekharaiah (1986) as $\tau_0 = 0.75 \times 10^{-13}$ sec and $\tau_1 = 0.5 \times 10^{-13}$ sec.

Because the algebraic Eq. (11) contains all the information about the wave speed and angular frequency, and the roots are complex for all considered values of wave number, therefore the waves are attenuated in space. We can write $c^{-1} = v^{-1} + i\omega^{-1}q$, so that $p = R + iq$, where $R = \omega / v$, v and q are real numbers. Upon using the above relation in Eq. (17), the values of the wave speed (v) and the attenuation coefficient (q) for different modes of wave propagation can be obtained.

Table 1 Comparison of non-dimensional frequencies among the Generalized Theory (GL), Lord-Schulman Theory (LS) and Classical Theory (CT) of thermo-elasticities for clamped and unclamped boundaries of thermally insulated circular plate

Mode	Un clamped			clamped		
	LS	GL	CT	LS	GL	CT
1	0.1672	0.0765	0.0139	0.1508	0.1342	0.1152
2	0.3335	0.2719	0.0541	0.2255	0.1969	0.1564
3	0.5337	0.4977	0.1174	0.5773	0.3248	0.2444
4	0.8292	0.4385	0.1994	0.5941	0.5593	0.3487
5	1.1408	0.6952	0.2964	0.6303	0.8050	0.6584
6	1.4579	0.8714	0.4051	0.7070	0.8512	0.7551
7	1.7707	1.1350	0.6478	1.2007	1.0230	0.9038

Table 2 Comparison of non-dimensional frequencies among the Generalized Theory (GL), Lord-Schulman Theory (LS) and Classical Theory (CT) of thermo-elasticities for clamped and unclamped boundaries of isothermal circular plate

Mode	Un clamped			Clamped		
	LS	GL	CT	LS	GL	CT
1	0.1781	0.1336	0.0532	0.1084	0.1049	0.0259
2	0.5747	0.2736	0.1801	0.2130	0.1213	0.1702
3	0.6492	0.2928	0.2063	0.3220	0.2563	0.2950
4	0.5391	0.3727	0.3967	0.3295	0.3732	0.3837
5	1.7853	0.4036	0.5010	0.4752	0.4831	0.5129
6	1.9288	0.5308	0.6400	0.6349	0.6422	0.8727
7	2.0824	0.7015	0.9025	0.9142	0.8231	0.9308

A comparison is made for the non-dimensional frequencies among the Generalized Theory (GL), Lord-Schulman Theory (LS) and Classical Theory (CT) of thermo-elasticity for the clamped and unclamped boundaries of the thermally insulated and isothermal circular plate in Tables 1 and 2, respectively. From these tables, it is clear that as the sequential number of the vibration modes increases, the nondimensional frequencies also increases for both the clamped and unclamped cases. Also, it is clear that the nondimensional frequency exhibits higher amplitudes for the LS theory compared with the GL and CT due to the combined effect of thermal relaxation times and damping of the foundation.

In Figs. 2 and 3, the dispersion of frequencies with the wave number is studied for both the thermally insulated and isothermal boundaries of the cylindrical plate in different modes of vibration. From Fig. 2, it is observed that the frequency increases exponentially with increasing wave number for thermally insulated modes of vibration. But smaller dispersion exist in the frequency in the current range of wave numbers in Fig. 3 for the isothermal mode due to the combined effect of damping and insulation.

In Fig. 4, the variation of attenuation coefficients with respect to the wave number of the cylindrical plate is presented for the thermally insulated boundary. The magnitude of the attenuation coefficient increases monotonically, attaining the maximum in $0.1 \leq \delta \leq 0.8$ for first four modes of vibration, and slashes down to become asymptotically linear in the remaining range of wave

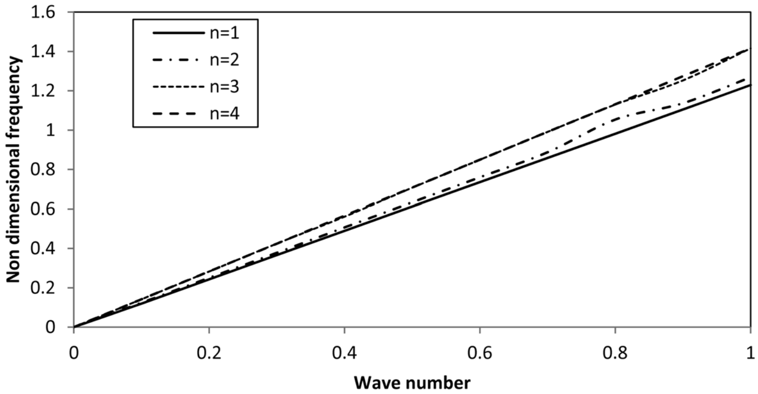

Fig. 2 Variation of nondimensional frequency of thermally insulated cylindrical plate with wave number on elastic foundation ($\nu = 0.3$, $K = 1.5 \times 10^7$, $p_2 = 0$)

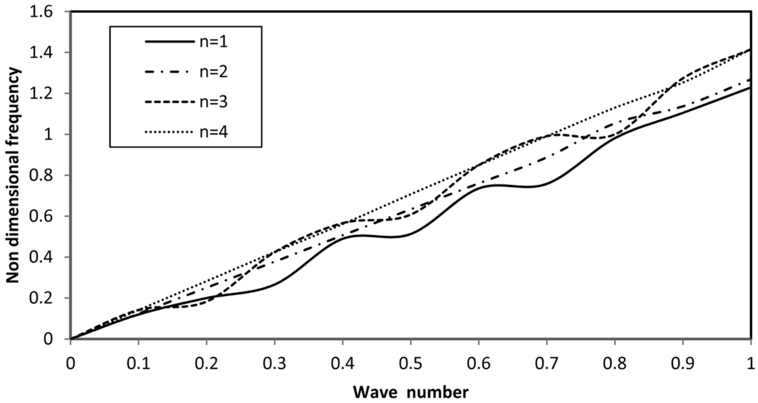

Fig. 3 Variation of nondimensional frequency of isothermal cylindrical plate with wave number on elastic foundation ($\nu = 0.3$, $K = 1.5 \times 10^7$, $p_2 = 0$)

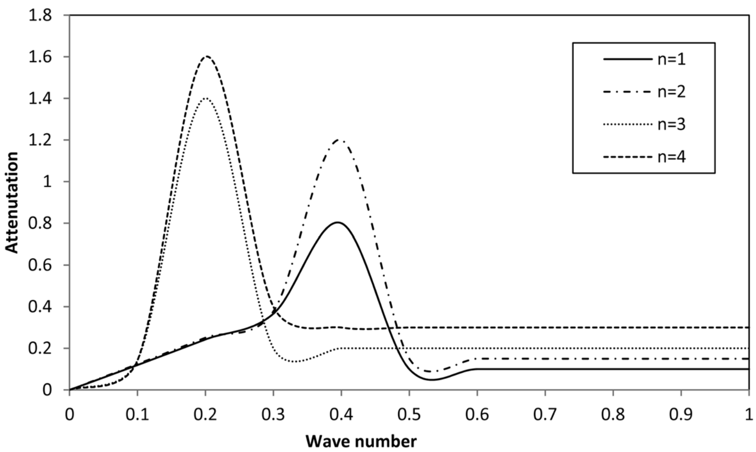

Fig. 4 Variation of attenuation of thermally insulated cylindrical plate with wave number on elastic foundation ($\nu = 0.3$, $K = 1.5 \times 10^7$, $p_2 = 0$)

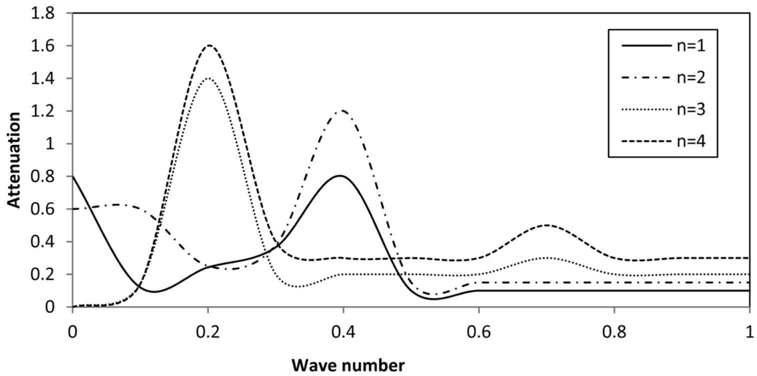

Fig. 5 Variation of attenuation of isothermal cylindrical plate with wave number on elastic Foundation ($\nu = 0.3$, $K = 1.5 \times 10^7$, $p_2 = 0$)

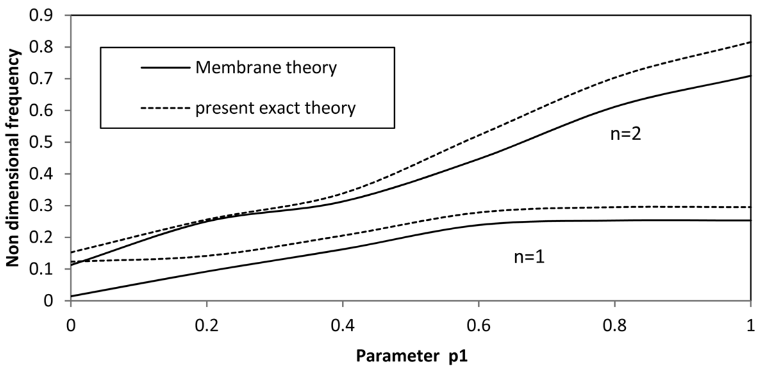

Fig. 6 Variation of the foundation parameter p_1 versus Non-dimensional frequency ($\nu = 0.3$, $p_2 = 0$)

number. The variation of attenuation coefficients with respect to the wave number of the isothermal cylindrical plate is presented in Fig. 5, where the attenuation coefficient attains the maximum in $0.1 \le \delta \le 0.8$ with a small oscillation in the starting wave number, and decreases to become linear due to the relaxation times. From Figs. 4 and 5, it is clear that the effects of stress free thermally insulated and isothermal boundaries of the plate are quite pertinent due to the combined effect of thermal relaxation times and damping effect of the foundation.

Fig. 6 reveals that the variation of non dimensional frequency with the foundation parameter p_1 for the first and second modes without the thermal effect. The exact theory is compared with the membrane theory by Paliwal (1955), it is clear that the present exact theory agrees well with the membrane theory for the first and second modes. This is identical to the well-known property of the membrane theory for the uncoupled problem. However, for the thinner panel, when the effect of the foundation is obvious, the frequency of membrane becomes smaller than the exact one. From the comparison of the dispersion curves in Fig. 4, it is quite clear that due to the damping effect of the foundation on the outer sides of the plate, the non dimensional frequency varies significantly and becomes steady for $p_1 \ge 0.5$. The dispersion curves become smoother in this case than those in the absence of foundation parameter because of the shock absorption nature of the foundation.

6. Conclusions

The two dimensional wave propagation of a homogeneous isotropic generalized thermo elastic cylindrical plate embedded on the Winkler-type elastic foundation was investigated in this paper. For this problem, the governing equations of three dimensional linear theory of generalized thermo elasticity have been employed in the context of the Lord and Schulman theory and solved by the modified Bessel function with complex arguments. The effects of the frequency and attenuation coefficient with respect to the wave number and the foundation parameter p_1 on the natural frequencies of a closed Zinc cylindrical plate was investigated, with the results presented as the dispersion curves. By comparing the present results with those of the membrane theory by Paliwal (1955), it is clear that the present exact theory with respect to the foundation agrees well with the membrane theory. In addition, a comparative study is made among the LS, GL and CT theories and the frequency change is observed to be highest for the LS theory, followed by the GL and CT theories due to the thermal relaxation effects and damping.

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